On Control Problems with Bounded State Variables

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Abstract: **In a recent paper [1] we showed, among other things, how a fairly general control problem, or programming problem, with constraints can be reduced to a special type of Bolza problem in the calculus of variations. Necessary conditions for the Bolza problem were then translated into necessary conditions for optimal control. These conditions include the maximum principle of Pontryagin [2, 3] for this class of problems and some of the later results of Gamkrelidze [4]. Our results in [1] do not, however, apply to control problems with constraints on the state variables that do not explicitly involve the control variable. Such problems were treated by Gamkrelidze in [4], who modified the arguments in [2] to account for the additional constraints. In this memorandum we shall use the methods of [1] to study such problems, and we shall obtain the results of Gamkrelidze, with one exception, from relevant results in the calculus of variations.**

Keywords: **Optimal Control, Constraints, Peacewise smooth boundary, Modification, Interior, Restriction.**

1. FORMULATION OF PROBLEM

We assume that the reader is familiar with [1], and we shall use the notation of [1]. Let θ be a function of class C'' on the region $\mathscr{D}$ of (t,x)-space such that the relation $\theta(t,x)=0$ defines a manifold $\mathscr{B}^*$ which divides $\mathscr{D}$ into two regions. Let $\mathscr{B}$ be that subset of $\mathscr{D}$ defined by the relation

$$\theta(t,x)\geq 0. \qquad (2.1)$$

We shall consider the control problem as formulated in [1] with $g\equiv 0$, and with the added restriction that the curves *K* resulting from controls *u* in $\mathscr{A}$ must lie in $\mathscr{B}$. That is, we consider the following problem.

Problem I. Find an element u^* in the class of admissible controls $\mathscr{A}$ that minimizes the functional

$$J(u)=\int_{t_0}^{t_1} f(t,x,u)\,dt.$$

Here, the state of the system is determined by the system of differential equations

$$x'=G(t,x,u), \quad x(t_0)=x_0;$$

the controls and the state of the system satisfy the constraints

$$R(t,x,u)\geq 0, \quad \theta(t,x)\geq 0;$$

and the right-hand end point (t_1, x_1) of the trajectory is a point of a *p*-dimensional manifold, $0 \le p \le n$.

It will be clear from what follows and from [1] how one handles the case in which $\mathscr{B}$ is a closed region in $\mathscr{D}$ with piecewise smooth boundary, each piece of which is defined by a relation $\theta_i(t,x)=0$.

The assumption that $g \equiv 0$ is made in order to simplify a certain portion of the argument below. Now loss of generality will result, for the control problem of [1] is equivalent to a control problem in which $g \equiv 0$ and the initial point lies on a line, as the following transformation shows. Let a new coordinate x^{n+1} be introduced by means of the following differential equation and end conditions:

$$\frac{dx^{n+1}}{dt}=0,\ x_1^{n+1}=\frac{g(\sigma)}{t_1(\sigma)-t_0},\quad x_0^{n+1}\ \text{free}.$$

Let the functional to be minimized be

$$J(u)=\int_{t_0}^{t_1}\left(f+x^{n+1}\right)dt.$$

In Problem I we assumed, as we did in[1], that the left-hand end point (t_0, x_0) is fixed. If (t_0, x_0) is constrained to lie on a *p*-dimensional manifold $(1 \le p \le n)$ in (t,x)-space, the analysis that follows requires the introduction of a transversality condition for the left-hand end point. We leave this to the reader.

We cannot treat Problem I by simply adjoining the constraint (2.1) as an $(r+1)$-st component to the constraint vector $R(t,x,u)$ and then proceeding to use the analysis of [1]. The reason is that since θ is independent of *u*, the constraint vector (R,θ) will not satisfy the constraint condition (2.2)-(ii) of [1] at any point of $\mathscr{B}^*$, the manifold defined by $\theta(t,x)=0$. The constraint conditions for Problem I are the following, which are modifications of those in [1]:

(i) If $r > m$, then at each point of $\mathscr{S} = \mathscr{B} \times \mathscr{U}$, where $\mathscr{B}^0$ denotes the interior of $\mathscr{B}$, at most *m* components of *R* can vanish. If $r \ge m$, then at each point of $\mathscr{S}^* = \mathscr{B}^* \times \mathscr{U}$ at most $(m-1)$ components of *R* can vanish.

(ii) At each point of $\mathscr{S}$ the matrix $\left(\partial R^i / \partial u^j\right)$, where i ranges over those indices such that $R^i(t,x,u)=0$, and $j=1,\ldots,m$, has maximum rank. At each point of $\mathscr{S}^*$, if the *m*-dimensional row vector $\theta_x G_u$ is adjoined to this matrix (where *i* now ranges from 1 to *m* – 1), the resulting matrix has maximum rank. (2.2)

2. EQUIVALENT LAGRANGE PROBLEM

Let η be a scalar, and let

$$\gamma(t,x,\eta)=\begin{cases}\eta^4-\theta(t,x) & \text{if } \eta \ge 0,\\ \theta(t,x) & \text{if } \eta \le 0.\end{cases} \tag{3.1}$$

The function $\gamma(t,x,\eta)$ is clearly C'' on the region of (t,x,η)-space which is the Cartesian product of the η-axis and $\mathscr{D}$. If we let

$$y'=u,\quad y(t_0)=0,$$

then Problem I is readily seen to be equivalent to the following problem of Lagrange in $(n+m+r+2)$-dimensional (t,x,y,ξ,η) -space.

Problem II. Find an arc $(x(t),y(t),\xi(t),\eta(t))$ that minimizes

$$\int_{t_0}^{t_1} f(t,x,y')\,dt \tag{3.2}$$

in the class of arcs that are piecewise C'' and that satisfy the differential equations

$$\begin{aligned} &G(t,x,y')-x'=0, \\ &R(t,x,y')-(\xi')^2=0, \\ &\theta_t+\theta_x x'-\gamma_\eta \eta'=0, \end{aligned} \tag{3.3}$$

and also the end conditions

$$x(t_0)=x_0, y(t_0)=0, \xi(t_0)=0, \gamma(t_0,x_0,\eta_0)=0, \tag{3.4}$$

$$t_1=t_1(\sigma), x_1=x_1(\sigma), \gamma(t_1,x_1,\eta_1)=0, \tag{3.5}$$

$$\xi_1 \equiv \xi(t_1), \text{free}, \ y_1 \equiv y(t_1), \text{free}.$$

Note that the last equation in (3.3) and the end conditions imply that $\gamma(t,x,\eta)=0$ along each arc. Hence, by (3.1), $\theta \geq 0$ along each arc.

Let us suppose that u^* is an optimal control in $\mathscr{A}$, and let K^* be the corresponding curve in (t,x)-space. Let K_1^* be the corresponding arc in (t,x,y,ξ,η) -space. Consider the following $(n+r+1)$ by $(m+n+r+1)$ matrix along K_1^* :

$$\left\| \begin{matrix} G_{Y'}-I & 0 & 0 & 0 \\ R_{y'} & 0 & -2\Xi' & 0 \\ 0 & \theta_x & 0 & -\gamma_\eta \end{matrix} \right\|. \tag{3.6}$$

where $2\Xi'$ is an $r \times r$ diagonal matrix with entries $2(\xi^i)'$ on the diagonal. The rank of (3.6) is clearly the same as the rank of the matrix

$$\left\| \begin{matrix} G_{y'} & -I & 0 & 0 \\ R_{y'} & 0 & -2\Xi' & 0 \\ \theta_x G_{y'} & 0 & 0 & -\gamma_\eta \end{matrix} \right\|. \tag{3.7}$$

Since $\gamma=0$ along K_1^*, it follows from (3.1) that $\gamma_\eta>0$ if $\theta>0$, and that $\gamma_\eta=0$ if $\eta \leq 0$. Hence, using the constraint conditions (2.2) and arguments similar to those used to determine the rank of (3.8) in [1], we see that the matrix (3.7) has rank $(n+r+1)$ at all points of K_1^*. Hence (3.6) has rank $(n+r+1)$.

The above argument is not restricted to K_1^*; it shows that (3.6) has rank $(n+r+1)$ at all elements $(t,x,y,\xi,\eta,x',\xi',\eta')$ of a curve for which (3.3) and the end condition (3.4) and (3.5) hold.

If $t_1(\sigma)$ and $x_1(\sigma)$ define a p-dimensional terminal manifold for Problem I, then the right-hand end conditions for Problem II and the restriction that (t_1,x_1) is in $\mathscr{B}$ determine a p-dimensional terminal manifold in (t,x,η)-space for Problem II. We suppose that in a neighbourhood of the right-hand end points of K_1^*, this manifold is given parametrically as follows:

$$t_1 = t_1(s), \qquad x_1 = x_1(s), \qquad \eta_1 = \eta_1(s).$$

We note that

$$\gamma\big(t_1(s), x_1(s), \eta_1(s)\big) \equiv 0. \tag{3.8}$$

Since u^* is an optimal control, it is clear that K_1^* furnishes a minimum for Problem II. From this and from the preceding discussion it follows that the multiplier rule, the Weierstrass condition, and the Clebsch condition as given in Bliss [5] and extended by McShane [6] hold along K_1^*. The function F in this instance is defined as follows:

$$F = \psi_0 f + \psi(G - x') + \mu\Big(R - (\xi')^2\Big) + v\big(\theta_t + \theta_x x' - \gamma_\eta \eta'\big). \tag{3.9}$$

From the Euler equations

$$\frac{dF_{y'}}{dt} = F_y, \quad \frac{dF_{\xi'}}{dt} = F_\xi,$$

the continuity of $F_{y'}$ and $F_{\xi'}$, and the relations

$$F_{y'} = 0, \qquad F_{\xi'} = 0 \qquad \text{at} \qquad (t_1, x_1, \eta_1), \tag{3.10}$$

which we obtain from the transversality condition, we deduce as we did in [1] that along K_1^*,

$$F_{\xi_i} = \mu^i R^i = 0, \quad i = 1,\dots,r,$$

$$F_{y'} = \psi_0 f_{v'} + \psi G_{v'} + \mu R_{v'} = 0. \tag{3.11}$$

From the Euler equation $dF_{\eta'}/dt = F_\eta$ we find that between corners of K_1^*,

$$v'\gamma_\eta = 0. \tag{3.12}$$

The Euler equation $dF_{x'}/dt = F_x$ takes the form

$$\frac{d}{dt}\left(-\psi + v\theta_x\right) = \psi_0 f_x + \psi G_x + \mu R_z + v\left(\theta_{tx} + \theta_{xx} x'\right). \tag{3.13}$$

If we make use of (3.10) in the remaining equations of the transversality condition, we get

$$\left(F - x'F_{x'} - \eta' F_{\eta'}\right)t_{1s} + F_{x'}x_{1s} + F_{\eta'}\eta_{1s} = 0.$$

From (3.3) we obtain that $x' = G$ and that $F = \psi_0 f$ at (t_1, x_1, η_1). Using these relations, Eq. (3.4), and the last equation in (3.3), we can rewrite the preceding equation as follows:

$$(\psi_0 f + \psi G) t_{1s} - \psi x_{1s} + v(\theta_t t_{1s} + \theta_x x_{1s} - \gamma_\eta \eta_{1s}) = 0.$$

Hence, using (3.8) we find that the transversality condition, in addition to yielding (3.10), gives

$$(\psi_0 f + \psi G) t_{1s} - \psi x_{1s} = 0. \tag{3.14}$$

Another necessary condition is the continuity along K_1^* of the expression

$$F - x'F_{x'} - y'F_{y'} - \xi' F_{\xi'} - \eta' F_{\eta'}.$$

If we take (3.3) and (3.10) into account, this expression may be rewritten as follows along K_1^*:

$$\psi_0 f + \psi G + v\theta_t. \tag{3.15}$$

Using arguments similar to those used above and in [1], we may rewrite the Weierstrass condition in the following form:

$$\psi_0 \left(f(t, x, Y') - f(t, x, y') \right) + \psi (X' - x') \geq 0. \tag{3.16}$$

From the Clebsch condition we deduce, as we did in [1], that along K_1^*,

$$\mu \leq 0, \tag{3.17}$$

and that

$$e\left((\psi_0 f + \psi G + \mu R)_{y'y'} \right) e \geq 0 \tag{3.18}$$

for all *m*-dimensional solution vectors *e* of the following linear systems: $\hat{R}_{y'} e = 0$ at points that are interior to $\mathscr{B}$; $\hat{R}_{y'} e = 0$ and $\theta_x G_{y'} e = 0$ at points of K_1^* that correspond to points of $\mathscr{B}^*$. The vector $\hat{R}$ is obtained from *R* by taking those components of *R* that vanish at the point.

3. AN INTERIOR SEGMENT

We now consider the curve K^* corresponding to the optimal control u^*. We adopt Gamkrelidze's definition [4] and say that the point $(\tau, x(\tau))$ of K^* is a junction point if it belongs to K^*, if $t_0 < \tau < t_1$ and if there exists a $\delta > 0$ such that either the segment of K^* for which $\tau - \delta < t < \tau$, or the segment for which $\tau < t < \tau + \delta$ (or both), lies in the interior of $\mathscr{B}$. We call τ a junction time. We suppose that K^* has a finite number of junction points. For definiteness, we suppose that if τ is the largest junction time, then the portion of K^* defined for $\tau < t < t_1$ is interior to $\mathscr{B}$. We denote this segment by K_A^* and also use this notation for the corresponding segment of K_1^*.

Since K_A^* is interior to $\mathscr{B}$, it follows that $\eta > 0$. Hence from (3.1) we have $\gamma_\eta > 0$, and so from (3.12) we get that between corners of K_A^*, v is constant. Moreover, the end condition (3.14) places no restriction on $v(t_1)$. Since *v* is constant between corners of K_1^*, equation (3.13) can be written as

$$\psi' = -(\psi_0 f_x + \psi G_x + \mu R_x). \tag{4.1}$$

Note that the differential equation (4.1) and the end conditions (3.14) are independent of the values assigned to *v*. Using this observation and the fact that the left-hand end point of K_1^* is fixed, we can see by examining the proof in [6] that we can always choose $(\psi_0, \psi, \mu) \neq 0$ at the right-hand end-point (t_1, x_1) of K_1^*. From the constraint condition (2.2)-(ii) and from (3.11) it follows that *l* is determined uniquely as a linear function of (ψ_0, ψ) on the interval $\tau \leq t \leq t_1$. Hence, on this interval, (4.1) can be written as a linear differential equation in ψ, and so if $(\psi_0, \psi) = 0$ at any point of K_A^*, then $(\psi_0, \psi) \equiv 0$. Moreover, in this event $\mu \equiv 0$. Since $(\psi_0, \psi, \mu) \neq 0$ at (t_1, x_1), we can therefore conclude that $(\psi_0, \psi) \neq 0$ at every point of K_A^*.

Define

$$H(t, x, u, \lambda_0, \lambda) = \lambda_0 f + \lambda G. \tag{4.2}$$

For $\tau < t < t_1$, let

$$\lambda_0 = \psi_0, \qquad \lambda = \psi, \tag{4.3}$$

and take *v* to be a constant on the entire interval $\tau < t < t_1$. (K_1^* may have corners in this interval.) It follows from (4.1)-(4.3), (3.11), (3.14)-(3.18), and from the relation $y' = u$, that Theorem 2 of [1] holds along K_A^*.

4. A BOUNDARY SEGMENT

Let τ' be the largest of the junction times that are less than τ. If there are none, take $\tau' = t_0$. We next suppose that the segment of K^* defined for $\tau' \leq t \leq \tau$ lies entirely in $\mathscr{B}^*$. We denote this segment (and the corresponding segment of K_1^*) by K_B^*. To simplify the exposition we shall suppose that K_B^* has no corners and the same components of *R* vanish all along K_B^*. If the contrary holds, the argument requires trivial modifications, which we leave to the reader.

Let $$\phi(t, x, u) = \theta_i + \theta_x G. \tag{5.1}$$

Then along K_B^*, we clearly have

$$\phi = 0. \tag{5.2}$$

On the interval $\tau' \leq t \leq \tau$, let

$$\lambda_0 = \psi_0, \quad \lambda = \psi - v\theta_x. \tag{5.3}$$

If we substitute (5.3) into (3.13) and use the definition of ϕ given in (5.1) and the definition of *H* given in (4.2, we can rewrite (3.13) as

$$\frac{d\lambda}{dt} = -\left(H_x + \mu R_x + v\phi_x\right). \tag{5.4}$$

If we substitute (5.3) into the second equation of (3.11) and replace y' by *u*, we can rewrite this equation as follows:

$$H_u + \mu R_u + v\theta_x G_u = 0. \tag{5.5}$$

Similarly, the substitution of (5.3) into the Weierstrass condition (3.16) leads to the relation

$$H(t, x, Y') - H(t, x, y') + v\theta_x (X' - x') \geq 0$$

for all admissible (t,x,X',Y'). Since the element (t,x,X',Y') is admissible, it satisfies the last equation of (3.3), and so $\theta_x(X'-x')=\theta_i-\theta_i=0$. Hence, setting $u=y'$, we can rewrite (3.16) as

$$H(t,x,u)\geq H(t,x,u^*) \tag{5.6}$$

along K_B^* for all admissible *u* such that $\phi(t,x,u)=0$.

Finally, equation (3.18) of the Clebsch condition becomes

$$e\left((H+\mu R+v\theta_{iG})_{uu}\right)e\geq 0. \tag{5.7}$$

From the necessary conditions for Problem II, it follows that there exists a constant $\lambda_0\geq 0$ and functions (λ,μ,v) such that (5.4)-(5.7) hold along K_B^*. From the constraint condition (2.2)-(ii), it follows that we may solve (5.5) uniquely for (μ,v) as linear functions of (λ_0,λ). Substitution of this solution into (5.4) yields a system of linear differential equations for λ. Hence if (λ_0,λ) is determined at a point of K_B^*, then the solution λ of (5.4) and the functions (μ,v) are uniquely determined along K_B^*.

From the Weierstrass-Erdmann corner condition for Problem II, we see that

$$F_{x'}=-\psi+v\theta_x \tag{5.8}$$

is continuous at the junction point $(\tau,x(\tau))$. Thus, if we denote functions along K_A^* by a subscript *A* and functions along K_B^* by a subscript B_i then at $(\tau,x(\tau))$ we have

$$-\psi_A^+ + v_A^+\theta_x = -\psi_B^- + v_B^-\theta_x. \tag{5.9}$$

Using (4.3) and (5.3), this can be rewritten as

$$\lambda_B^- = \lambda_A^+ - v_A^+\theta_x,\quad v_A^+ \text{ arbitrary.} \tag{5.10}$$

This relation and the relation

$$\lambda_{0B}=\lambda_{0A}=\psi_0, \tag{5.11}$$

which follows from (4.3), (5.3) and the constancy of ψ_0, therefore serve to determine $(\lambda_0,\lambda,\mu,v)$ uniquely along K_B^*.

From the continuity of (3.15), and from (4.3), (5.3), and (5.2), we readily find that at the junction point $(\tau,x(\tau))$,

$$H^- = H^+ + v_A\theta_t. \tag{5.12}$$

With the help of (5.10) and (5.2), this can be rewritten as follows:

$$\lambda_0(f^- - f^+)+\lambda^*(G^- - G^+)=0,$$

$$\lambda_0(f^- - f^+)+\lambda^-(G^- - G^+)=v_A^+(\theta_x G^+ + \theta_t). \tag{5.13}$$

It is readily verified that the vector

$$(\lambda_0,\lambda_B,\mu_B,v_B)=(0,\rho\theta_x,0,-\rho), \tag{5.14}$$

where ρ is any real constant, satisfies (5.4) and (5.5) along any curve K obtained from an admissible control and lying in $\mathscr{B}^*$. Substitutions of (5.14) into the relation (5.6) reduces (5.6) o the identity $\rho\theta_t = \rho\theta_t$ along any such curve. From (5.5) and the constraint condition (2.2)-(ii) it follows that (5.14) is the unique solution of (5.4)-(5.6) with $\lambda_0 = 0, \lambda = \rho\theta_x$, ρ arbitrary. It is immediate from (5.3) that for Problem II,

$$(\psi_0, \psi_B, \mu_B, \nu_B) = (0,0,0,-\rho), \quad \rho \text{ arbitrary,} \tag{5.15}$$

is the unique vector corresponding to (5.14). The vector (5.15) reduces the Euler equations (3.13) and (3.11), and the Weierstrass condition (3.16) to identities for all admissible curves lying in $\mathscr{B}^*$. We shall refer to (5.14) or (5.15) as a trivial multiplier vector. Note that since ρ is arbitrary, the zero vector is included.

If we have

$$\lambda_0^+ = 0, \quad \lambda_A^+ = k\theta_x, \quad k \neq 0, \tag{5.16}$$

at $(\tau, x(\tau))$, for every set of multipliers $(\lambda_0, \lambda_A, \mu_A, \nu_A)$ such that Eqs. (5.4)-(5.7) hold along K_A^*, then it follows from (5.10) and the discussion of the preceding paragraph that along the segment K_B^* we obtain the trivial multipliers (5.14). To avoid this, we proceed as follows. We consider a Problem I', which we define as Problem I with fixed initial point $(\tau', x(\tau'))$ and fixed terminal point $(\tau, x(\tau))$. It is clear that the segment K_B^* must furnish a relative minimum for Problem I'. For, if some other curve K' furnished a minimum, then we could replace the segment K_B^* of K^* by K' and thereby contradict the minimalityof K^*. Since the Lagrange problem corresponding to Problem I' has fixed end points, it can be seen from the proof in [6] of the necessary conditions that we may take $(\psi_0, \psi(\tau_2)) \neq 0$. Hence there exist nontrivial multipliers (ψ_0, ψ, μ, ν) such that (5.4)-(5.7) hold along K_B^*, even if (5.16) holds. In this event, however, (5.10)-(5.13) are no longer valid.

The following theorem is a consequence of the preceding discussion.

Theorem 1. On the interval $[\tau', \tau]$ there exists a constant $\lambda_0 \geq 0$ and a continuous *n*-dimensional vector $\lambda(t)$ such that $(\lambda_0, \lambda) \neq (0, \rho\theta_x)$, ρ arbitrary; an *r*-dimensional vector $\mu(t) \leq 0$, continuous except perhaps at values of t corresponding to corners of K_B^*; and a function $\nu(t)$ with the same continuity properties as μ, such that along K_B^*, (5.4)-(5.7) hold.

At the junction point $(\tau, x(\tau))$, either (5.16) holds for every $(\lambda_{0A}, \lambda_A^+)$, or the jump conditions (5.10) and (5.12) (and hence (5.11)) hold.

Remark 1. This result was obtained by Gamkrelidze [4], who used different arguments. He also presents a result that in our notation reads $d\nu / dt \geq 0$ along K_B^*.

Remark 2. In Section 6 we assumed that the segment K_A^* was interior to $\mathscr{B}$. If we had assumed that the segment K_A^* was in $\mathscr{B}^*$, then we would still conclude that, except for the jump conditions, Theorem 1 holds along K_A^*.This follows from the arguments used to establish the theorem and the observation that the transversality condition (3.14) places no restriction on $\nu(t_1)$.

5. CONCLUSION

If $\tau' > t_0$, let us suppose that τ'' is the largest junction time that is less than τ'; if there is no such junction time, take $\tau'' = t_0$. The segment of K^* defined for $\tau'' < t < \tau'$ then lies entirely within $\mathscr{B}$. We denote his segment by K_C^*. On this segment we define (λ_0, λ) by means of (4.3), and apply the analysis of Section 6, except for the determination of initial data for (4.1). We now use the continuity of (5.8) at the junction point to determine λ_{0C} and $\lambda_C(\tau'')$, where the subscript C refers to functions along K_C^*. Since we have $(\lambda_0, \lambda_B^+) \neq (0, \rho\theta_x)$, it follows that at $(\tau'', x(\tau''))$, the following jump condition holds:

$$\lambda_C^- = \lambda_B^+ + v_C^- \theta_x \neq 0, \qquad v_C^- \text{ arbitrary.}$$

Hence the conclusions of Theorem 2 of [1] hold along K_C^*.

Let us now suppose that K_B^*, instead of lying on the boundary as assumed in Section 5, is an interior segment. The point $(\tau, x(\tau))$, however, is still assumed to be a junction point. Along K_B^* we now define $(\lambda_{0B}, \lambda_B)$ by (4.3), and we apply the analysis of Section 6, except for the determination of the initial data for $(\lambda_{0B}, \lambda_B)$. To determine λ_{0B} *and* $\lambda_B(\tau)$ we use the continuity of (5.8) and get

$$\lambda_B^- = \lambda_A^+ + \left(v_B^- - v_A^+\right)\theta_x .$$

Since we are free in our choice of v_B^-, we get

$$\lambda_B^- = \lambda_A^+ + k\theta_x \neq 0 . \tag{6.1}$$

Hence Theorem 2 of [1] holds along K_B^*.

Note that in obtaining (6.1), we did not make use of the special fact that v_A^+ is arbitrary because the segment K_A^* terminates at (t_1, x_1). Moreover, we can choose v_B^- so that $k \geq 0$.

We summarize the principal results of this paper in the following theorem.

Theorem 2. Let $u^* \in \mathscr{A}$ be an optimal control, and let K^* be the corresponding curve. Then there exists a constant $\lambda_0 \geq 0$, an *n*-dimensional vector $\lambda(t)$, an *r*-dimensional vector $\mu(t) \leq 0$, and a function $v(t)$ such that the following hold: Along a segment of K^* whose end points are junction points and that is in the interior of $\mathscr{B}$, except for the end-points, Theorem 2 of [1] holds. Along a segment whose end-points are junction points and that lies in $\mathscr{B}^*$, Theorem 1 of this paper holds. If $\lambda_0 = 0$, then at a junction point either

$$\lambda^- + k\theta_x = 0, \qquad k \neq 0, \tag{6.2}$$

or

$$\lambda^+ = \lambda^- + k\theta_x \neq 0 . \tag{6.3}$$

If $\lambda_0 \neq 0$, then at a junction point (6.3) holds. At a junction point between two interior segments, we may take *k* so that (6.2) does not occur, and $k \geq 0$ in (6.3). If (6.3) holds, then the following also holds:

$$H^+ = H^- - k\theta_t . \tag{6.4}$$

From this theorem, several observations can be made. Since these are given by Gamkrelidze [4], there is no need to repeat them here.

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